Lecture 8

ASSIGNMENT

READ 47-53 "FORWARD / INVERSE SLR TRANSFORM"

TODAY

ROTATIONS

3 x 3 ORTHONORMAL MATRICES (SO(3))

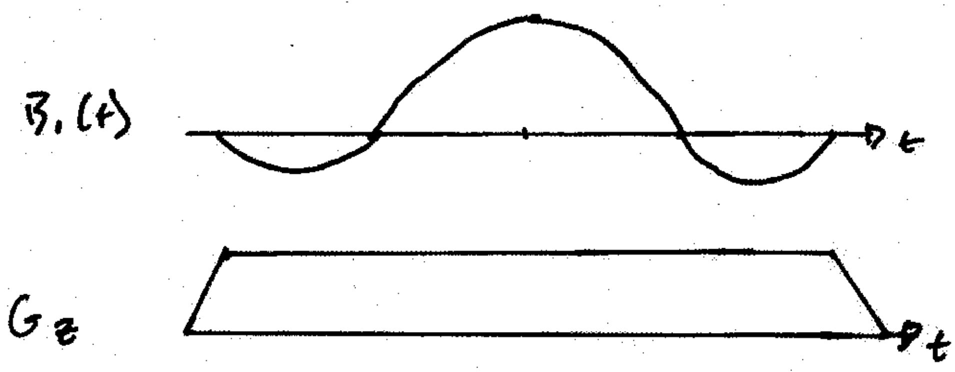
2 x 2 UNITARY MATRICES (SU(2))

SPINORS

SPIN DOMAIN

LARGE-TIP-ANGLE SOLUTIONS FOR THE BLOCH EQN

GIVEN A LARGE-TIP-ANGLE SLICE SELECTIVE PULSE



AND SOME INITIAL MAGNETIZATION

$$\underline{M}_0 = (m_{x,0}, m_{y,0}, M_{z0}),$$

WHAT IS THE MAGNETIZATION AFTER THE PULSE?

SEVERAL QUANTITIES OF INTEREST:

INITIAL MAGNETIZATION $\underline{M}_0 = (0, 0, m_0)$

$m_x + i m_y$ EXCITATION PROFILE

m_z INVERSION / SATURATION PROFILE

INITIAL MAGNETIZATION $\underline{M}_0 = (0, m_0, 0)$

$m_x + i m_y$ SPIN-ECHO PROFILE

MOTION OF THE MAGNETIZATION GOVERNED BY BLOCH EQUATION

$$\frac{d}{dt}\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} = \begin{pmatrix} 0 & \gamma G x & -\gamma B_{1,y} \\ -\gamma G x & 0 & \gamma B_{1,x} \\ \gamma B_{1,y} & -\gamma B_{1,x} & 0 \end{pmatrix}\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$$

NEGLECTING T_1, T_2 AND ASSUME WE ARE EXACTLY ON RESONANCE

WE CAN WRITE THIS MORE COMPACTLY USING THE SPIN MATRICES

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$S_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{S} = (S_x, S_y, S_z)$$ VECTOR OF MATRICES

THEN

$$\frac{d}{dt}\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} = \left[-\gamma B_{1,x}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \gamma B_{1,y}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \gamma Gx \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$$

$$\frac{d}{dt}\underline{m} = \left[(-\gamma B_{1,x}, -\gamma B_{1,y}, -\gamma Gx) \cdot \underline{S} \right] \underline{m}$$

DEFINE

$$\omega = -\gamma \sqrt{B_{1,x}^2 + B_{1,y}^2 + (Gx)^2}$$ RATE OF ROTATION

$$\underline{n} = \frac{\gamma}{|\omega|}(B_{1,x}, B_{1,y}, Gx)$$ AXIS OF ROTATION

WE WILL ASSUME

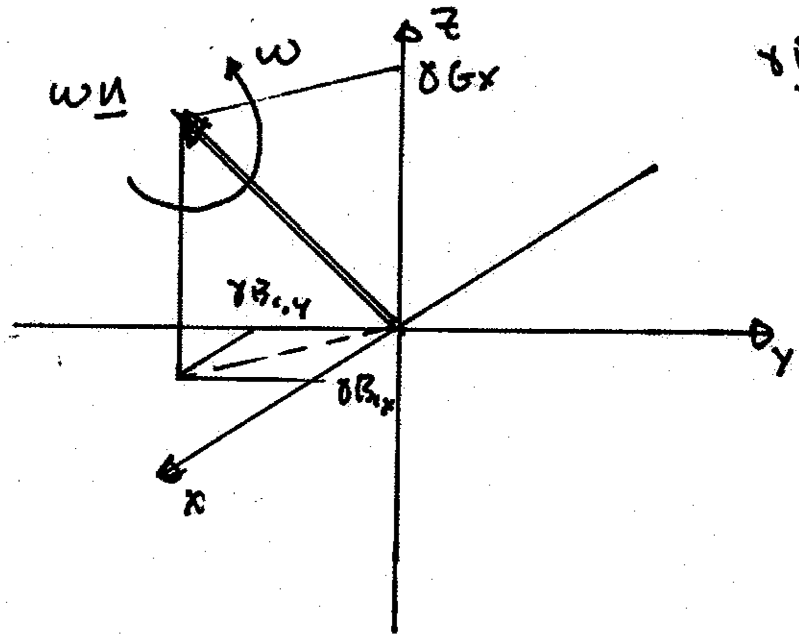
G IS CONSTANT

$B_{1,x}, B_{1,y}$ ARE TIME VARYING

HENCE

$\omega, \underline{n}$ ARE TIME VARYING

THE $(-\gamma)$ IS DUE TO THE FACT THAT PROTONS PRECESS IN THE LEFT-HAND SENSE, AND THE SPIN MATRICES ARE RIGHT-HAND SENSE.



$\gamma \underline{B} = \gamma(B_{1,x}, B_{1,y}, Gx)$
$= -\omega \underline{n}$

MAGNETIZATION ROTATES ABOUT $\underline{n}$ AT A RATE ω

THEN

$$\frac{d}{dt} \underline{m} = \omega(\underline{n} \cdot \underline{S}) \underline{m}$$

SOLUTION WILL BE OF THE FORM

$$\underline{m}(\tau) = R\, \underline{m}(0)$$

WHERE R IS A 3×3 ORTHONORMAL MATRIX

NOT OBVIOUS HOW TO FIND R
(WE'LL SEE A FEW SPECIAL CASES LATER...)

NOTE THAT

$$R = e^{\int_{-\infty}^{t} \omega(\tau)\,(\underline{n}(\tau)\cdot\underline{S})\,d\tau}$$

IS NOT IN GENERAL A SOLUTION. THIS IS BECAUSE

$$e^{A+B} \neq e^{A} e^{B}$$

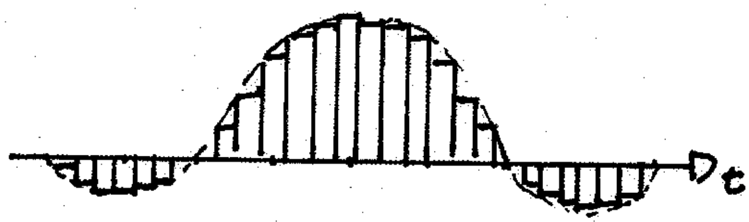
UNLESS A AND B COMMUTE ($AB = BA$)

ONE CASE WHERE IT DOES HOLD IS $\underline{n}$ CONSTANT.

$$R = e^{(\underline{n}\cdot\underline{S})\int_{-\infty}^{t}\omega(\tau)\,d\tau}$$

THIS WILL BE AN IMPORTANT SPECIAL CASE LATER.

PIECEWISE CONSTANT APPROXIMATION



RF MADE UP OF SHORT RECTANGLES, EACH PRODUCING A FLIP ANGLE

$$\gamma B_1(t_i)\,\Delta t$$

WHERE Δt IS THE WIDTH OF THE RECTANGLE

THE ROTATION PRODUCED IS

$$R_i = e^{(\underline{n}_i \cdot \underline{S})\,\omega_i \Delta t}$$

WHICH WE CAN SOLVE FOR AS A 3×3 ORTHONORMAL MATRIX

THE TOTAL ROTATION IS THEN

$$R = R_n R_{n-1} \cdots R_2 R_1$$

EXAMPLE: LET $\underline{n} = (1,0,0)$, A ROTATION ABOUT X, BY AN ANGLE θ

$$R = e^{S_x \theta}$$

$$= I + (\theta S_x) + \frac{1}{2}(\theta S_x)^2 + \frac{1}{6}(\theta S_x^3)$$

AFTER SOME TEDIOUS CALCULATION

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

GENERAL SOLUTION ARBITRARY $\underline{n}$, θ

$$R = e^{(\underline{n}\cdot\underline{S})\theta}$$

$$= I\cos\theta + (n^T n)(1-\cos\theta) + (\underline{n}\cdot\underline{S})\sin\theta$$

THIS IS THE SO(3) REPRESENTATION OF ROTATIONS

3×3 ORTHONORMAL MATRICES

GENERALLY TOO CUMBERSOME TO DO BY HAND!

SIMPLER REPRESENTATION

WE CAN ALSO REPRESENT ROTATIONS USING THE 2x2 UNITARY MATRICES

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

LET

$$\underline{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

THE DIFFERENTIAL EQUATION THAT CORRESPONDS TO THE BLOCH EQUATION IS

$$\dot{\psi} = \frac{i\omega}{2}(\underline{n}\cdot\underline{\sigma})\psi$$

WHERE $\underline{n}$ AND ω ARE THE SAME AS BEFORE

$$\omega = -\gamma\sqrt{B_{1,x}^2 + B_{1,y}^2 + (Gx)^2}$$

$$\underline{n} = \frac{\gamma}{|\omega|}(B_{1,x}, B_{1,y}, Gx)$$

ψ IS A SPINOR, WHICH WE WILL COME BACK TO.

FOR THE CASE WHERE $\underline{n}$ AND ω ARE CONSTANT (ONE SAMPLE OF PIECE-WISE CONSTANT PULSE)

DEFINE

$$\theta = \omega \Delta t$$

SOLUTION IS

$$\psi_{i+1} = Q\,\psi_i$$

WHERE

$$Q = e^{i\frac{\theta}{2}(\underline{n}\cdot\underline{\sigma})}$$

$$= I\cos\frac{\theta}{2} - i(\underline{n}\cdot\underline{\sigma})\sin\frac{\theta}{2}$$

SO

$$Q = \overbrace{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}^{I}\cos\theta/2 + \overbrace{\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}}^{\sigma_x}\left(-i n_x \sin\frac{\theta}{2}\right)$$

$$+ \underbrace{\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}}_{\sigma_y}\left(-i n_y \sin\theta/2\right) + \underbrace{\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}}_{\sigma_z}\left(-i n_z \sin\theta/2\right)$$

$$= \begin{pmatrix}\cos\theta/2 - i n_z \sin\theta/2 & -i(n_x - i n_y)\sin\theta/2\\ -i(n_x + i n_y)\sin\theta/2 & \cos\theta/2 + i n_z \sin\theta/2\end{pmatrix}$$

DEFINE

$$\boxed{\begin{aligned}\alpha &= \cos\theta/2 - i n_z \sin\theta/2\\ \beta &= -i(n_x + i n_y)\sin\theta/2\end{aligned}}$$

THEN

$$\boxed{Q = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}}$$

TWO COMPLEX NUMBERS DETERMINE ROTATION!

α AND β ARE THE CAYLEY-KLEIN PARAMETERS

ONE ADDITIONAL CONSTRAINT IS

$$\alpha\alpha^* + \beta\beta^* = 1$$

HENCE, THERE ARE ONLY 3 FREE PARAMETERS.

FOR OUR RF PULSE, THE TOTAL ROTATION IS

$$Q = Q_n Q_{n-1} \cdots Q_2 Q_1$$

PRODUCT OF 2x2 UNITARY MATRICES

SU(2) REPRESENTATION OF ROTATIONS.

HOWEVER, IT IS ACTUALLY EVEN SIMPLER.

LET

$$Q_n = \begin{pmatrix} a_n & -b_n^* \\ b_n & a_n^* \end{pmatrix}$$

BE ONE OF THE INCREMENTAL ROTATIONS, AND

$$\begin{pmatrix} \alpha_n & -\beta_n^* \\ \beta_n & \alpha_n^* \end{pmatrix} = \prod_{j=1}^{n} \begin{pmatrix} a_j & -b_j^* \\ b_j & a_j^* \end{pmatrix}$$

THEN

$$\begin{pmatrix} \alpha_n & -\beta_n^* \\ \beta_n & \alpha_n^* \end{pmatrix} = \begin{pmatrix} a_n & -b_n^* \\ b_n & a_n^* \end{pmatrix} \cdots \underbrace{\begin{pmatrix} a_j & -b_j^* \\ b_j & a_j^* \end{pmatrix} \cdots \begin{pmatrix} a_1 & -b_1^* \\ b_1 & a_1^* \end{pmatrix}}_{\begin{pmatrix} \alpha_j & -\beta_j^* \\ \beta_j & \alpha_j^* \end{pmatrix}}$$

WE REALLY ONLY NEED TO KEEP TRACK OF $(\alpha_j \ \ \beta_j)^T$

$$\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} = \begin{pmatrix} a_j & -b_j^* \\ b_j & a_j^* \end{pmatrix} \begin{pmatrix} \alpha_{j-1} \\ \beta_{j-1} \end{pmatrix}$$

PROPOGATE α, β BY 2X2 VECTOR MATRIX PRODUCTS.

THE VECTOR $(\alpha_i \ \beta_i)^T$ IS A SPINOR

$$\psi_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}$$

THE INITIAL CONDITION IS NO ROTATION ($\theta=0$)

SO

$$\psi_0 = \begin{pmatrix} \cos\theta/2 - i n_z \sin\theta/2 \\ -i(n_x + i n_y)\sin\theta/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

HALF ANGLES

THE SPINOR ROTATIONS ARE ALL HALF ANGLES!

A ROTATION BY 2π GIVES

$$\psi(2\pi) = \begin{pmatrix} \cos 2\pi/2 - i n_z \sin 2\pi/2 \\ -i(n_x + i n_y)\sin 2\pi/2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

ABOUT ANY $\underline{n}$. THE SPINOR CHANGES SIGN.

A ROTATION BY 4π GIVES

$$\psi(4\pi) = \begin{pmatrix} \cos(4\pi/2) - i n_z \sin 4\pi/2 \\ -i(n_x + i n_y)\sin 4\pi/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

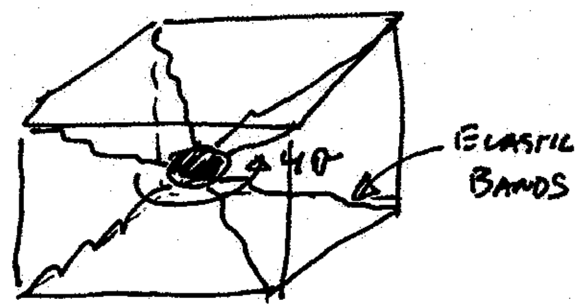
SO 720° IS THE IDENTITY ROTATION.

WHERE DOES THIS COME FROM?

PHYSICS: $\frac{1}{2}$ INTEGER SPIN PARTICLES (FERMIONS)
WAVE FUNCTION CHANGE SIGN WITH 2π ROTATION

EXTENDED OBJECT ROTATION:

OBJECT CONNECTED TO A FRAME BY ELASTIC BANDS



A 2π ROTATION CANNOT BE UNTANGLED

A 4π ROTATION CAN!

IMPLICATIONS FOR PULSE DESIGN

- MOST PULSES ARE BETWEEN 0 AND π

THEN

$\cos \theta/2$ GOES FROM 1 TO 0

$\sin \theta/2$ GOES FROM 0 TO 1

COMPLETELY UNAMBIGUOUS! NO PHASE UNWRAPPING PROBLEMS

VERY CONVENIENT

CHANGING DOMAINS

MAGNETIZATION TO SPIN DOMAIN

$$\omega = -\gamma \sqrt{B_{1,x}^2 + B_{1,y}^2 + (Gx)^2}$$

$$\underline{n} = \frac{\gamma}{|\omega|} \left(B_{1,x}, B_{1,y}, Gx \right)$$

$$\theta = \omega \Delta t$$

THEN $(\underline{n}, \theta)$ DETERMINE (α, β).

SPIN DOMAIN TO MAGNETIZATION

FOR A GIVEN SPINOR $\psi = (\alpha, \beta)^T$, THE (OBSERVABLE) MAGNETIZATION COMPONENTS ARE

$$m_x = \psi^* \sigma_x \psi \qquad m_y = \psi^* \sigma_y \psi \qquad m_z = \psi^* \sigma_z \psi$$

WHERE

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

AND

$$\psi = Q \psi_0 = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}$$

IF THE MAGNETIZATION IS INITIALLY ALONG +Z AXIS, $\theta=0$ AND

$$\psi_0 = \begin{pmatrix} \cos\frac{\theta}{2} - i n_z \sin\theta/2 \\ -i(n_x + i n_y)\sin\theta/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

THEN

$$\psi = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

WE CAN THEN COMPUTE

$$m_x = (\alpha^* \ \beta^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= (\alpha^* \ \beta^*)\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

$$= \underline{\alpha^*\beta + \beta^*\alpha}$$

$$m_y = (\alpha^* \ \beta^*)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= (\alpha^* \ \beta^*)\begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix}$$

$$= \underline{-i\alpha^*\beta + i\alpha\beta^*}$$

$$m_z = (\alpha^* \ \ \beta^*)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= (\alpha^* \ \ \beta^*)\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$$

$$= \underline{\alpha\alpha^* - \beta\beta^*}$$

RECALL THAT

$$\alpha\alpha^* + \beta\beta^* = 1$$

SO

$$m_z = (1 - \beta\beta^*) - \beta\beta^*$$

$$= \underline{\underline{1 - 2\beta\beta^*}}$$

AS USUAL, WE WILL BE INTERESTED IN

$$m_{xy} = m_x + i\, m_y$$

SUBSTITUTING FOR m_x AND m_y

$$m_{xy} = (\alpha^*\beta + \beta^*\alpha) + i(-i\alpha^*\beta + i\alpha\beta^*)$$

$$= \alpha^*\beta + \cancel{\beta^*\alpha} + \alpha^*\beta - \cancel{\alpha\beta^*}$$

$$= \underline{\underline{2\alpha^*\beta}}$$

WE COULD HAVE OBTAINED THIS DIRECTLY BY DEFINING

$$\sigma_{xy} = \sigma_x + i\,\sigma_y$$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= 2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ "RAISING OPERATOR"

THEN

$$m_{xy} = \psi^* \sigma_{xy} \psi$$
$$= (\alpha^* \ \beta^*) \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$
$$= (\alpha^* \ \beta^*) \begin{pmatrix} 2\beta \\ 0 \end{pmatrix}$$
$$= \underline{\underline{2\,\alpha^* \beta}}$$

WE CAN COMPUTE THE VARIOUS TERMS FOR m_{xy} AND m_{xy}^*, AND COLLECT THEM IN A MATRIX

$$\begin{pmatrix} m_{xy}^+ \\ m_{xy}^{+*} \\ m_z^+ \end{pmatrix} = \begin{pmatrix} (\alpha^*)^2 & -\beta^2 & 2\alpha^*\beta \\ -(\beta^*)^2 & \alpha^2 & 2\alpha\beta^* \\ -\alpha^*\beta^* & -\alpha\beta & \alpha\alpha^*-\beta\beta^* \end{pmatrix} \begin{pmatrix} m_{xy}^- \\ m_{xy}^{-*} \\ m_z^- \end{pmatrix}$$

FOR ANY INITIAL MAGNETIZATION $\underline{m}^-$ THERE IS A SIMPLE EXPRESSION FOR $\underline{m}^+$

<u>IMPORTANT SPECIAL CASES</u>

<u>EXCITATION PROFILE</u>, $\underline{m}^- = (0, 0, m_0)$

$$\underline{m_{xy}^+ = 2\alpha^*\beta\, m_0}$$

<u>INVERSION / SATURATION PROFILE</u>

$$\underline{m_z^+ = (\alpha\alpha^* - \beta\beta^*)\, m_0}$$

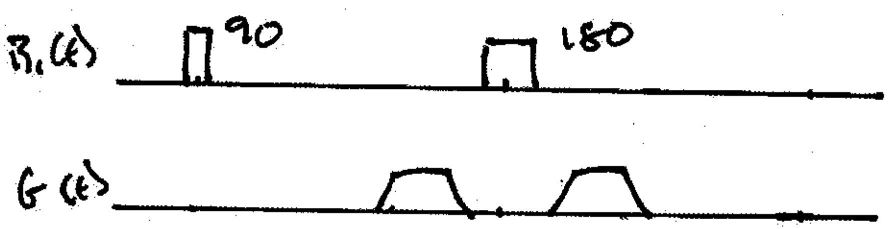
SPIN-ECHO PROFILE, $\underline{m}^- = (m_{xy}, m_{xy}^*, 0)$

$$m_{xy}^+ = (\alpha^*)^2 m_{xy}^- - \beta^2 m_{xy}^{-*}$$

IF THE INITIAL MAGNETIZATION IS ALONG +y (FOLLOWING A -90_x), $m_{xy} = i m_0$ AND

$$m_{xy}^+ = i\left((\alpha^*)^2 + \beta^2\right) m_0$$

TO IDENTIFY THE TWO TERMS, IT IS USEFUL TO CONSIDER THE FOLLOWING PULSE SEQUENCE



"CRUSHERS"

THE PHASE PRODUCED BY THE GRADIENT IS

$$\phi(x) = \left[-\gamma \int_{\text{LOBE}} G(\tau)\, d\tau\right]\cdot x$$

THE ROTATION IT PRODUCES IS

$$Q_c = \begin{pmatrix} e^{-i\phi(x)/2} & 0 \\ 0 & e^{i\phi(x)/2}\end{pmatrix}$$

THE ROTATION PRODUCED BY THE 180-CRUSHER COMPLEX IS

$$Q_c Q_{180} Q_c = \begin{pmatrix} e^{-i\phi(x)/2} & 0 \\ 0 & e^{i\phi(x)/2} \end{pmatrix} \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \begin{pmatrix} e^{-i\phi(x)/2} & 0 \\ 0 & e^{i\phi(x)/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\phi(x)/2} & 0 \\ 0 & e^{i\phi(x)/2} \end{pmatrix} \begin{pmatrix} \alpha e^{-i\phi(x)/2} & -\beta^* e^{i\phi(x)/2} \\ \beta e^{-i\phi(x)/2} & \alpha^* e^{i\phi(x)/2} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha e^{-i\phi(x)} & -\beta^* \\ \beta & \alpha^* e^{i\phi(x)} \end{pmatrix}$$

THE SPIN-ECHO PROFILE IS

$$m_{xy}^{+} = \underbrace{(\alpha^*)^2 e^{i2\phi(x)} m_{xy}^{-}}_{\text{CRUSHED TERM}} - \underbrace{(\beta)^2 m_{xy}^{-*}}_{\text{REFOCUSED TERM}}$$

THE FIRST TERM IS THE "CRUSHED" TERM OR "STRAIGHT THROUGH" TERM THAT IS NOT REFOCUSED BY THE SPIN-ECHO PULSE

THE SECOND TERM IS THE REFOCUSED, OR SPIN ECHO, TERM

HENCE

$$\underline{\underline{m^{+}_{xy,cR} = \left(\alpha^{*}\right)^{2} m^{-}_{xy}}}$$ (STRAIGHT THROUGH)

AND

$$\underline{\underline{m^{+}_{xy,SE} = -\beta^{2}\, m^{-\,*}_{xy}}}$$ (SPIN-ECHO)